Sufficient Conditions for the Existence of Optimum Beam Patterns for Unequally Spaced Linear Arrays with an Example

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Abstract—Dolph's method for determining the optimum element currents for half-wavelength equispaced discrete linear arrays is generalized to symmetric discrete linear arrays. The theorem proved gives sufficient conditions for the existence of optimum beam patterns for arrays with elements symmetrically positioned about the array center, but with fixed unequal spacings between the elements. The conditions are such that the Remes exchange algorithm for minimax approximation of functions can be employed to compute the optimum element currents corresponding to an optimum beam pattern directly from the given spacings of the elements. Half-wavelength spaced linear arrays satisfy the conditions of the theorem; therefore, it provides a new method of calculating the well-known Dolph–Chebyshev element currents. An example with unequal spacings is included to show the utility of the method even when the hypotheses of the theorem may not be met.

I. INTRODUCTION

Optimum beam patterns and element currents for single frequency linear arrays with a finite number of omnidirectional half-wavelength spaced elements were determined by Dolph [1] through a technique involving the Chebyshev polynomials. All these beam patterns have equal amplitude sidelobes. Sufficient conditions are given here for symmetric linear arrays to possess optimum beam patterns with equal amplitude sidelobes. This feature is precisely the fact needed in the calculation of the element currents.

The definition of an optimum beam pattern used in Dolph's paper will be used: a beam pattern is optimum if, for a given main lobe beamwidth, the overall sidelobe amplitude is minimized. Beamwidth is measured from the maximum response axis to the first null. The linear arrays considered in this paper are those whose elements are symmetrically spaced and have symmetrically tapered element currents about the center of the array.

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II. PRELIMINARIES

Let $f(x)$ be a real valued continuous function defined on the closed interval $[a,b]$. The norm of $f(x)$, denoted $\|f\|_{[a,b]}$, is defined to be

$$\|f\|_{[a,b]} = \max_{a\leq x\leq b} |f(x)|.$$

Now let $h_1(x),\cdots,h_N(x)$ be a given finite collection of real valued continuous functions defined on the closed interval $[a,b]$. The linear span of these basis functions is a proper closed subspace of the space of all continuous functions on the interval $[a,b]$ equipped with this max norm. It is known that there exist real constants $\alpha_1,\cdots,\alpha_N$ such that

$$\|f(x) - \sum_{i=1}^{N} \alpha_i h_i(x)\|_{[a,b]}$$

is a minimum. The function $h(x) = \sum_{i=1}^{N} \alpha_i h_i(x)$ is defined to be a minimiax approximation to the function $f(x)$ from the basis $h_1(x),\cdots,h_N(x)$. The crucial property that these basis functions must satisfy to guarantee the uniqueness of a minimax approximation is embodied in the definition: the functions $h_1(x),\cdots,h_N(x)$ form a Chebyshev basis of degree N on the closed interval $[a,b]$, if and only if every nontrivial linear combination of these functions possesses at most $N-1$ real roots in the interval $[a,b]$. A particularly well-known Chebyshev basis is the collection $1,x,\cdots,x^{N-1}$ on any finite or infinite interval. It is possible that a given collection of functions may be a Chebyshev basis on one interval but not on another. It can be shown that the functions $h_1(x),\cdots,h_N(x)$ form a Chebyshev basis on the interval $[a,b]$, if and only if the determinants

$$U(x_1,\cdots,x_N) = \begin{vmatrix} h_1(x_1) & h_1(x_2) & \cdots & h_1(x_N) \\ h_2(x_1) & h_2(x_2) & \cdots & h_2(x_N) \\ \vdots & & & \\ h_N(x_1) & h_N(x_2) & \cdots & h_N(x_N) \end{vmatrix} \neq 0$$

for all points x_i, such that

$$a \leq x_1 < x_2 < \cdots < x_{N-1} < x_N \leq b.$$

The reader is referred to Karlin and Studden [2] for a proof of this and other equivalent formulations of a Chebyshev basis, as well as for a proof of the following fundamental theorem.

Theorem

Let $h_1(x),\cdots,h_N(x)$ be a Chebyshev basis on the interval $[a,b]$. Then $h(x) = \sum_{i=1}^{N} \alpha_i h_i(x)$, for some real constants α_i, is a minimax approximation to $f(x)$ on $[a,b]$, if and only if there exist at least $N+1$ points $a \leq x_0 < x_1 < \cdots < x_M \leq b$, $M > N$, such that

$$f(x_i) - h(x_i) = \pm\|f-h\|_{[a,b]}, \qquad i = 0,\cdots,M$$

and the sign of the error alternates from point to point. Furthermore, the approximating function $h(x)$ is unique.

In other words there exists exactly one linear combination of the N Chebyshev basis functions which has at least $N+1$ points of equiripple (but alternately signed) error for a given function $f(x)$, and it is this linear combination that forms the unique minimax approximation to $f(x)$. If the basis functions do not form a Chebyshev basis, however, the minimax error curve need not be equiripple.

The Remes exchange algorithm [3] employs the equal oscillation error of the minimax approximation to compute the constants $\alpha_1,\cdots,\alpha_N$. The algorithm is iterative and has been shown to converge under very general conditions.

III. THE SUFFICIENCY THEOREM

As stated in the introduction, every linear array considered is assumed to be symmetric and to have symmetrically tapered element currents about the array's center. If the center of an M element linear array is chosen as the origin of the coordinate system, then the field pattern, as a function of the angle measured from a normal to the array, is proportional to the absolute value of

$$\sum_{i=1}^{N} \alpha_i \cos\left(\frac{2\pi x_i}{\lambda}\sin\theta\right), \qquad 0 \le \theta \le 2\pi$$

where

$$N = \left[\frac{M+1}{2}\right]$$

λ wavelength of design frequency

x_i distance of ith element (counted from the center of the array)

α_i current of the elements at x_i (if M is odd, α_1 is half the current).

Putting $u = \pi\sin\theta$ and restricting θ to $0 \le \theta \le \pi/2$ to utilize symmetry, the field pattern is proportional to the absolute value of

$$P(u) = \sum_{i=1}^{N} \alpha_i \cos(\xi_i u), \qquad 0 \le u \le \pi \tag{1}$$

where $\xi_i = x_i/(\lambda/2)$, $i = 1,\cdots,N$. We always have $0 \le \xi_1 < \xi_2 < \cdots \le \xi_N$. For the field pattern $P(u)$ defined in (1), we define the sidelobe level on the interval $[u_1,\pi)$ to be the norm $\| P(u) \|_{[u_1,\pi)}$, where $u_1 > 0$ is the first null of $P(u)$. We define the sidelobe ratio on the same interval to be the ratio $| P(0) | \div \| P(u) \|_{[u_1,\pi)}$. Note that both these terms are in linear units. Also, the symbol for the half-open interval $[a,b)$ is interpreted to mean the closed interval $[a, b-\epsilon]$, where ϵ is some preselected small positive number. We now state and prove the main result.

Sufficiency Theorem

Suppose that the functions

$$\cos(\xi_1 u),\cdots,\cos(\xi_N u) \tag{2}$$

form a Chebyshev basis on the interval $[0,\pi)$, and that the functions

$$\cos(\xi_1 u),\cdots,\cos(\xi_{N-1} u) \tag{3}$$

form a Chebyshev basis on the interval $[u_0,\pi)$ for some real number u_0, $0 \le u_0 < \pi$. Then there is an angle θ_0, $0 \le \theta_0 < \pi/2$, such that for any specified beamwidth θ, $\theta_0 \le \theta < \pi/2$, there exists a unique optimum field pattern. This optimum field pattern will have equal amplitude sidelobes.

Proof: Since the functions (3) form a Chebyshev basis on the interval $[u_0,\pi)$, there must exist a unique minimax approximation to the function $-\cos(\xi_N u)$ from this basis; that is, there exist constants $\alpha_1,\cdots,\alpha_{N-1}$ such that

$$-\cos \xi_N u \doteq \alpha_1 \cos \xi_1 u + \cdots + \alpha_{N-1}\cos \xi_{N-1} u.$$

Thus if e_0 is the magnitude of the maximum error committed by this uniform approximation, then the function

$$f(u) \equiv \alpha_1 \cos \xi_1 u + \cdots + \alpha_{N-1}\cos \xi_{N-1} u + \cos \xi_N u$$

must oscillate about the zero function in the interval $[u_0,\pi)$ with the magnitude of the oscillation no greater than e_0 and with at least $(N-1)+1 = N$ points where the oscillation is exactly e_0. Let u_1 be the first zero of $f(u)$ greater than u_0 and let $\theta_0 = \sin^{-1}(u_1/\pi)$. We claim that $f(u)$ constitutes the optimum field pattern for a beamwidth to the first null of θ_0. For if this pattern is not optimum, there exists another field pattern function

$$g(u) = \beta_1 \cos \xi_1 u + \cdots + \beta_{N-1}\cos \xi_{N-1} u + \beta_N \cos \xi_N u$$

such that $g(0) = f(0)$, u_1 is the smallest positive root of $g(u)$, and g has a strictly smaller sidelobe level on the interval $[u_1,\pi)$. Since $f(u)$ has at least N points of maximum error on $[u_0,\pi)$, $f(u)$ has at least $N-1$ points of maximum error on $[u_1,\pi)$. It is clear that any function which is constrained to agree with f at $u = u_1$, which is everywhere strictly less than f on $[u_1,\pi)$, must intersect f in at least $N-1$ points in the interval $[u_1,\pi)$. Additionally, $f(0) = g(0)$, so that f and g must agree with at least N points in $[0,\pi)$. However, $f(u) - g(u)$ is then a linear combination of the functions (2), which has at least N zeros on $[0,\pi)$, contradicting the definition of a Chebyshev basis unless $f = g$. Thus f is the unique optimum field pattern for a beamwidth of θ_0.

To complete the proof of the theorem, we need to demonstrate that for each angle $\theta > \theta_0$, there exists a number $\hat{u}_0$, $u_0 < \hat{u}_0 < \pi$, such that the function

$$f(u) = \gamma_1 \cos \xi_1 u + \cdots + \gamma_{N-1}\cos \xi_{N-1} u + \cos \xi_N u$$

has $\hat{u}_1 = \pi\sin\theta$ as its first real root greater than or equal to $\hat{u}_0$, where $\sum_{i=1}^{N-1}\gamma_i \cos \xi_i u$ is the uniform approximation to $-\cos \xi_N u$ on $[\hat{u}_0,\pi)$. Since $[\hat{u}_0,\pi) \subset [u_0,\pi)$, the collection of functions (3) must form a Chebyshev basis on all intervals $[\hat{u}_0,\pi)$, so that the function $f(u)$ is well-defined for each $\hat{u}_0$. Also $\hat{u}_0 > u_0$ implies that $\hat{u}_1 > u_1$ (u_1 as defined earlier) since otherwise the beam pattern for a beamwidth of θ_0 is not optimum. Finally, as $\hat{u}_0$ is varied continuously, the constants $\gamma_1,\cdots,\gamma_{N-1}$ vary continuously, so that the first real zero greater than $\hat{u}_0$ varies continuously. Since $\hat{u}_0$ may be taken as close to π as desired, it must be the case that for some $\hat{u}_0$ the first real zero of $f(u)$ greater than $\hat{u}_0$ is equal to $\pi\sin\theta$.

IV. DOLPH–CHEBYSHEV SHADINGS AS A SPECIAL CASE

As mentioned in the introduction, the Dolph–Chebyshev shadings are designed specifically for a half-wavelength equispaced line array. We consider here only the case of $2N$, $N \ge 1$, elements in the array. For an odd number of elements, the arguments are essentially unchanged. Counting from the center of the array, the position x_i of the ith element is

$$x_i = \left(\frac{2i-1}{2}\right)\frac{\lambda}{2}, \qquad i = 1,\cdots,N$$

where λ is the wavelength of the design frequency. Then

$$\xi_i = \frac{2x_i}{\lambda} = \frac{(2i-1)}{2}$$

so that from (1), the field pattern is proportional to be absolute value of

$$P(u) = \sum_{i=1}^{N} \alpha_i \cos\left(\frac{2i-1}{2}u\right)$$

where α_i is the element current of the ith element from the center of the array. The Dolph–Chebyshev coefficients are determined for each specified beamwidth greater than zero. To apply the theorem of Section III, it is necessary to show that the N functions

$$\cos\left(\frac{u}{2}\right), \cos\left(\frac{3}{2}u\right),\cdots,\cos\left(\frac{2N-1}{2}u\right) \tag{4}$$

and the $N-1$ functions

$$\cos\left(\frac{u}{2}\right), \cos\left(\frac{3}{2}u\right),\cdots,\cos\left(\frac{2N-3}{2}u\right) \tag{5}$$

form Chebyshev bases on the intervals $[0,\pi)$ and $[u_0,\pi)$, where $u_0 = 0$ here. Consider, then, any linear combination of the functions (4)

$$f(u) = \sum_{i=1}^{N} \beta_i \cos\left(\frac{2i-1}{2}u\right), \qquad 0 \le u < \pi$$

$$= \sum_{i=1}^{N} \beta_i T_{2i-1}\left(\cos\frac{u}{2}\right)$$

TABLE I
OPTIMUM ELEMENT CURRENTS FOR FIELD PATTERNS GIVEN IN FIG. 1 WITH A COMPARISON TO OPTIMIZED EQUISPACED ARRAYS

u_0	.40138	.50138	.66138	.78138	.88138
Element Currents α_1	.66524	1.40005	2.60994	4.23857	5.99725
α_2	.56748	1.14597	2.03999	3.16892	4.32041
α_3	.41913	.79211	1.30817	1.88695	2.41420
α_4	.23500	.39087	.54719	.66452	.73742
α_5	1.00000	1.00000	1.00000	1.00000	1.00000
Side-Lobe Level (dB)	-9.92	-14.91	-20.11	-25.12	-29.50
Beamwidth (deg)	9.72	11.68	13.73	15.74	17.50
Beamwidth (deg) for 10-element, half-wavelength, equispaced array with Dolph-Chebyshev for the same side-lobe levels.	9.69	11.57	13.62	15.65	17.49

where T_{2i-1} is the Chebyshev polynomial of degree $2i - 1$. Put $u = 2\cos^{-1}(x)$. Then $0 < x \leq 1$ and

$$f(u) \equiv g(x) = \sum_{i=1}^{N} \beta_i T_{2i-1}(x)$$

so that $g(x)$ is a polynomial of degree at most $2N - 1$. Thus $g(x)$ can have at most $2N - 1$ zeros in any interval. Furthermore, $g(x)$ is an odd function and so can have at most $N - 1$ zeros in the interval $(0,1]$, so that $f(u)$ can have at most $N - 1$ zeros in $[0,\pi)$. Hence the functions (4) form a Chebyshev basis on the interval $[0,\pi)$. Replacing N by $N - 1$ in this argument shows that the functions (5) also form the required basis.

By the theorem of Section III, for each specified beamwidth, $\theta \geq 0$, there exists a unique optimum field pattern and a unique set of element currents. These currents are the Dolph–Chebyshev coefficients for the beamwidth θ. It should be pointed out that, if $0 \leq \theta < \pi/(4N - 2)$, the sidelobes have larger amplitude than the main lobe. The next section shows how the Remes exchange algorithm may be used as an alternative means of calculating the Dolph–Chebyshev coefficients, although the usual methods of calculation of these coefficients are preferable to this method.

V. EXAMPLE

The example chosen is a ten-element linear array with the elements located at positions proportional to the abscissas of a ten-point Gaussian quadrature formula:

$$\xi_1 = 0.68788$$
$$\xi_2 = 2.00253$$
$$\xi_3 = 3.13926$$
$$\xi_4 = 3.99708$$
$$\xi_5 = 4.50000.$$

The length of this array is the same as that of a ten-element half-wavelength equispaced array, but the element positions are substantially displaced from equal spacing. An effort to verify that the functions (2) and (3) form Chebyshev bases in this case was unsuccessful, and direct numerical verification was not attempted. Instead, the Remes exchange algorithm was employed immediately to find the element currents, and the observed behavior of the algorithm itself was used to make inferences about the functions (2) and (3). In this case, it will be seen that the functions (2) are not, in fact, a Chebyshev basis, and that it is likely that the functions (3) are not either. The example presented here shows the utility of the approach even when the hypotheses of the theorem of Section III do not apply.

Ma [4] describes what is essentially the Remes exchange algorithm and applies it to the synthesis of nonuniformly spaced arrays. However, Ma seeks approximations to the function $f(u) = \exp(-Au^2)$, where A is a positive real number, so that the element currents obtained are only approximately optimum. To find the optimum element currents, proceed as in the proof of the theorem in Section III to find a minimax approximation to $-\cos(\xi_5 u)$ in the form

$$-\cos(\xi_5 u) \doteq \sum_{k=1}^{4} \alpha_k \cos(\xi_k u) \tag{6}$$

on some interval $[u_0,\pi)$, $u_0 > 0$. The error curve of this approximation over the full interval $[0,\pi)$ is identically the optimum beam pattern for the beamwidth determined by the first null. The parameter u_0 alone controls the tradeoff between the sidelobe level and the beamwidth. Therefore, u_0 is varied systematically here.

To begin, the Dolph–Chebyshev coefficients corresponding to a sidelobe level of $R = -10$ dB were used as the initial guess for α_1, α_2, α_3, and α_4, so that the choice

$$u_0 = 2\cos^{-1}\left(\frac{1}{z_0}\right) \doteq 0.40138$$

was made, where

$$z_0 = \frac{1}{2}\left\{\left[r + (r^2 - 1)^{1/2}\right]^{1/L} + \left[r - (r^2 - 1)^{1/2}\right]^{1/L}\right\}$$

with $L = M - 1 = 9$ and $r = 10^{R/20}$. With this initial guess on this interval, the Remes exchange algorithm computed the minimax approximation (6) in two iterations and produced a result shown in Table I. To continue the procedure, u_0 was incremented by 0.01, and the Remes algorithm employed again using these newly computed coefficients as the initial guess on this smaller interval. Convergence occurred in two iterations, the beamwidth increased slightly, and the sidelobe level reduced to -10.3 dB. Continuing in this fashion, u_0 was systematically increased from 0.40138 to 1.02138. Representative beam patterns appear in Fig. 1 and the corresponding element currents in Table I. Notice that the beamwidths attainable by the Dolph–Chebyshev current amplitudes for an equispaced array are remarkably close to those obtained in this example.

By inspecting the beam patterns with the three lowest sidelobe levels, it is seen that each of these beam patterns possesses 5 zeros.

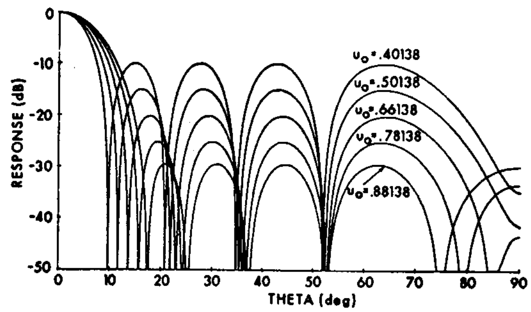

Fig. 1. Optimum field patterns for ten-element symmetrically positioned and unequally spaced linear array.

Since each of these beam patterns is also identically the error curve of a minimax approximation (6) on an interval $[u_0,\pi)$, it must be concluded that the functions (2) do not form a Chebyshev basis on the interval $[0,\pi)$. Consequently, the element currents may not be unique.

For each iteration of the Remes exchange algorithm, the solution of a system of linear equations is required. If there are $N - 1$ Chebyshev basis elements, then one equation in N unknowns is established for each point where equiripple error should occur. By the theorem in Section II, there must be at least $(N - 1) + 1 = N$ points of equiripple error. In the five results given for the present example, there are exactly $N = 5$ points of equiripple error, counting one point on the main lobe down at the sidelobe level (at $u = u_0$), so that unexpected numerical difficulties do not occur. To proceed further than these results requires the solution of six equations in five unknowns, because of the growth of the extra lobe at $\theta = 90°$. The straightforward procedure of solving any five of these six equations proved unsatisfactory because erratic behavior developed in the sidelobe corresponding to the equation deleted. An attempt to solve all six equations in both the least squares sense and the least maximum error sense by employing the generalized inverse of the coefficient matrix also proved unsatisfactory. It would seem, then, that either numerical difficulties are the cause of the problem or that the functions (3) do not form a Chebyshev basis. The author favors the latter possibility.

It should be noted that the Dolph–Chebyshev element currents for both a 10-element and a 50-element half-wavelength equispaced array have been computed in the aforementioned manner without difficulty from -10 dB to over -70 dB. (In these cases, an extra side lobe at 90° never develops.) Unequally spaced arrays with as many as 50 elements have also been successfully treated by this method.

All calculations were performed in double precision on the Univac 1108. Total CPU time, including plot generation for the example given was 67 s, although a more carefully written program could have reduced this time by at least a factor of two. A total of 63 sets of current amplitudes were computed.

VI. SUMMARY

Sufficient conditions for the existence of optimum field patterns for symmetrically spaced and amplitude tapered linear arrays have been proved. The theorem proved is a generalization of the work of Dolph on half-wavelength spaced linear arrays. A well-known algorithm from approximation theory has been employed in an example to compute the element currents corresponding to the optimum beam patterns using only the given element spacings themselves.

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