Real Excitation Coefficients Suffice for Sidelobe Control in a Linear Array

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Abstract—**Minimax design of a linear antenna array with arbitrary fixed elements leads to the following mathematical problem.**

$$\underset{w_k \text{ complex}}{\text{minimize}} \quad \max_{u_0 \le |u| \le u_1} |T(u)|$$

subject to $T(0) = 1$ where $T(u) = \Sigma_{k=1}^{N} w_k \exp(-id_k u)$ and d_k are real numbers. It is proven that this problem has a solution with *real* excitation coefficients w_k. In the antenna application this shows that there is no need to allow phasing at the individual elements of the array; amplitude control alone will achieve all the sidelobe reduction possible. An analogous result can be proved for a more general complex approximation problem.

We consider a linear antenna array with N omnidirectional elements located at arbitrary fixed positions $\{x_k\}$ receiving a plane wave of wavelength λ from the direction θ_a, $-\pi/2 \le \theta_a \le \pi/2$, relative to a normal to the array. If the array is steered to look in the direction θ_l, $-\pi/2 \le \theta_l \le \pi/2$, then the complex transfer function of the beamformer is given by

$$T(u) = \sum_{k=1}^{N} w_k \exp(-id_k u)$$

where $\{w_k\}$ are the element excitation coefficients, $d_k = 2\pi x_k/\lambda$, and $u = \sin\theta_a - \sin\theta_l$. The coefficients w_k may be complex in general. The peak response should occur at $u = 0$; we make the usual normalization

$$T(0) = \sum_{k=1}^{N} w_k = 1.$$

To effect small sidelobes we wish to minimize $|T(u)|$ for $|u| \ge u_0$ where $u_0 > 0$ is chosen small.

The total range of u depends on the look direction θ_l. First, let us consider only the case of the array steered broadside. Thus $\theta_l = 0$ and the range of u becomes $-1 \le u \le 1$ corresponding to $-1 \le \sin\theta_a \le 1$ for $-\pi/2 \le \theta_a \le \pi/2$. Hence the problem of selecting excitation coefficients to effect minimum overall sidelobe level becomes a minimax problem.

$$\underset{w_k \text{ complex}}{\text{minimize}} \quad \max_{u_0 \le |u| \le 1} |T(u)|$$

subject to

$$T(0) = \sum_{k=1}^{N} w_k = 1. \tag{1}$$

The case where the array is steered through the same number of degrees either side of broadside is very similar mathematically to the case $\theta_l = 0$ and is discussed below.

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A standard argument shows that a solution to problem (1) exists; however, it may not be unique. In general the excitation coefficients w_k are allowed to be complex; we now prove that a solution of (1) exists with w_k all *real*. First, denoting complex conjugates by an overbar,

$$\max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} \bar{w}_k e^{-id_k u} \right| = \max_{u_0 \le |u| \le 1} \left| \overline{\sum_{k=1}^{N} \bar{w}_k e^{-id_k u}} \right| = \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} w_k e^{id_k u} \right| = \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} w_k e^{-id_k u} \right|.$$

The last equality follows from the fact that $u_0 \le |-u| \le 1$ if and only if $u_0 \le |u| \le 1$; i.e., the range of u is symmetric about $u = 0$. Now

$$\begin{aligned}\max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} (\operatorname{Re} w_k) e^{-id_k u} \right| &= \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} \frac{1}{2} (w_k + \bar{w}_k) e^{-id_k u} \right| \\ &\le \tfrac{1}{2} \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} w_k e^{-id_k u} \right| + \tfrac{1}{2} \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} \bar{w}_k e^{-id_k u} \right| \\ &= \max_{u_0 \le |u| \le 1} \left| \sum_{k=1}^{N} w_k e^{-id_k u} \right| \text{ from above.}\end{aligned}$$

This guarantees the existence of a real solution of problem (1) as asserted, since

$$\sum_{k=1}^{N} w_k = 1 \text{ implies } \sum_{k=1}^{N} (\operatorname{Re} w_k) = \operatorname{Re} \sum_{k=1}^{N} w_k = 1.$$

We now note that, since $|T(-u)| = |\overline{T(u)}| = |T(u)|$ when $w_1, \cdots, w_N$ are real, we can further simplify problem (1) to

$$\underset{w_k \text{ real}}{\text{minimize}} \quad \max_{u_0 \le |u| \le 1} |T(u)| \tag{1a}$$

subject to

$$T(0) = \sum_{k=1}^{N} w_k = 1.$$

Hence, we can find a solution to problem (1) by solving the easier problem (1a). This has important practical implications for the design of an antenna array. It indicates that there is no need to allow phasing at the individual elements; amplitude of excitation alone will achieve all the sidelobe reduction that is possible.

The above analysis was for the look angle $\theta_l = 0$. Now let us regard θ_l as not being fixed; then the range of u becomes $-2 \le u \le 2$. The problem corresponding to (1) with θ_l bounded away from endfire is

$$\underset{w_k}{\text{minimize}} \max_{u_0 \le |u| \le 2-u_1} |T(u)|$$

subject to

$$T(0) = \sum_{k=1}^{N} w_k = 1. \tag{2}$$

As above, we can show the existence of a solution of (2) with *real* excitation coefficients w_k.

Now, let us consider a more general complex approximation problem. Let $f, h_1, \cdots, h_N$ be continuous complex valued functions defined on a closed and bounded set Q in the complex plane. (Q can be finite or infinite.) The minimax approximation problem is

$$\underset{a_k}{\text{minimize}} \max_{z \in Q} \left| f(z) - \sum_{k=1}^{N} a_k h_k(z) \right|. \tag{3}$$

Here the a_k are allowed to be complex. If for all $z \in Q$, $f(\bar{z}) = \overline{f(z)}$, $h_k(\bar{z}) = \overline{h_k(z)}$, $k = 1, \cdots, N$ and Q is symmetric with respect to the real axis, i.e., $\bar{q}$ in Q if and only if q in Q, then a solution of (3) with real coefficients exists. We omit the details of the verification.

Finally, we note that real excitation coefficients are not adequate for every use of a linear array or for every pattern desired. For example, if a null is required in the pattern at a point $\hat{u} \neq 0$, the equation $T(\hat{u}) = 0$ would be added to problem (1). However, now a solution with real coefficients would not necessarily exist.