Detection of Gaussian signals in Poisson-modulated interference

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Passive broadband detection of target signals by an array of hydrophones in the presence of multiple discrete interferers is analyzed under Gaussian statistics and low signal-to-noise ratio conditions. A nonhomogeneous Poisson-modulated interference process is used to model the ensemble of possible arrival directions of the discrete interferers. Closed-form expressions are derived for the recognition differential of the passive-sonar equation in the presence of Poisson-modulated interference. The interference-compensated recognition differential differs from the classical recognition differential by an additive positive term that depend on the interference-to-noise ratio, the directionality of the Poisson-modulated interference, and the array beam pattern. [S0001-4966(00)03910-2]

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I. INTRODUCTION

Passive broadband (incoherent) detection of a known Gaussian target signal in independent Gaussian noise by an array of hydrophones is a classical and well-understood problem. The processing chain of the optimal detector is depicted in Fig. 1. After time delaying and summing the outputs $\{u_m(t)\}$ of individual array hydrophones, the sum $v(t)$ passes through a predetection filter, the output $x(t)$ is squared to obtain $y(t)$, and then $y(t)$ is sent through a postdetection low-pass filter. The output $z(t)$ is compared to a threshold to determine the presence or absence of signal. The optimal predetection, or Eckart, filter maximizes the deflection coefficient, denoted by d, that parametrizes the receiver operating characteristic (ROC) curves of probability of detection, P_{D}, versus probability of false alarm, P_{FA}, when the variance of $z(t)$ remains approximately constant in the presence and absence of signal.

Interfering signals contribute coherently to the output of the array beamformer, and they inevitably degrade detection performance when present. The classic paper by Schultheiss[1] derives the deflection coefficient d for a Gaussian signal and a single interferer for the conventional processor in the case of an equispaced line array. Because there is only one interferer, he is able to discuss performance in terms of the behavior of d as a function of the angular separation between the signal and interferer. For more than one interferer, however, this approach is intractable because it requires examining all possible separations between signal and interferers.

When many interferers are present, it is reasonable to consider adopting a statistical model for both the number and arrival directions of the interferers. In this paper the expected values of the numerator and denominator of the expression for the deflection coefficient d are obtained in closed form under the assumption that the interference is Poisson modulated (PM). These explicit, but average, expressions facilitate insight into the problem while avoiding the otherwise intractable enumeration over all possible deterministic interference locations. PM interference models are used to study both conventional and optimal (Eckart) processors for a general hydrophone array. The PM interference model is especially well suited to study processors operating on linear beamformer outputs, but extensions to adaptive processors such as minimum variance distortionless response (MVDR) requires further work and are not considered here.

In some applications, the PM interference model can be obtained from a specified surface shipping density function $\mathrm{SD}(x,y)$. Thus, $\mathrm{SD}(x,y)$ is the probability that a surface ship is located at the point (x,y) at a randomly chosen time. Acoustic propagation transforms the given shipping density into a directional source distribution at the sensor, as depicted in Fig. 2 for the special case of straight-line propagation and a flat bottom with only two dominant paths, direct and bottom bounce. The directional source distribution at the sensor is denoted by $\Lambda(\xi)$, and it is in general determined by the propagation paths for nonconstant sound-speed profiles. In other applications, a volumetric source density might be employed to model biologics such as whales. Whatever the application, however, the investigation in this paper begins by assuming that the function $\Lambda(\xi)$ is given. Thus, the source distribution function $\Lambda(\xi)$ implicitly incorporates the effects of acoustic propagation.

PM interference is used to model both ''temporally nonresolvable'' and ''temporally resolvable'' interference. Temporally nonresolvable interference is essentially a shot noise model; that is, interferers are short duration and have a rapid occurrence rate compared to the averaging time T of the postdetection filter. The variance of the detector output is essentially stationary in this case and is directly related to the beamformer output variance. It is shown in Sec. III that there is essentially no difference between this kind of interference and classical nonisotropic noise. In contrast, temporally resolvable interference is long duration and has a slow occurrence rate compared to the averaging time T of the postdetection filter. In this case, the variance of the detector output varies over time, and the expected value of the variance of the detector output is computed with respect to the PM interference process. The expected variance involves the fourth power of the array beam pattern and is derived in Sec. IV.

Interference correction terms to the classical recognition differential of the passive-sonar equation are given in closed

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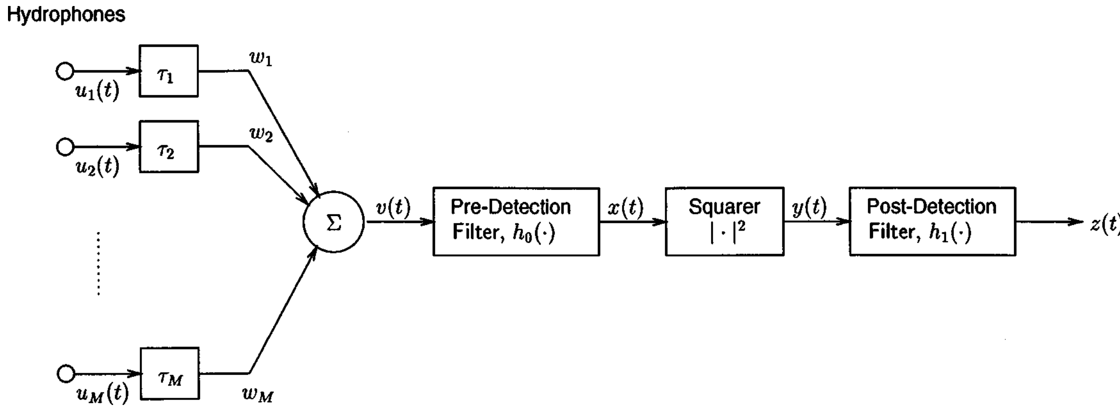

FIG. 1. Processing chain for detecting Gaussian signals in additive independent Gaussian noise.

form in Sec. V. These correction terms are derived directly from the deflection coefficient d of the conventional processor for fixed interference and for PM interference. [The quantity d is known by several other names, including the deflection signal-to-noise ratio (SNR), the detection coefficient, and simply the deflection.] Section VI summarizes the results obtained and presents some concluding remarks. Appendix A provides relevant background information on the Poisson process and its expectation operator. Appendix B discusses the deflection coefficient, the optimal predetection filter for Gaussian processes, and the Gaussian signals that are least detectable by the optimal processor.

Throughout the paper, target signal and (hydrophone) noise are assumed to be Gaussian random processes with stationary spectra. Noise is assumed statistically independent from hydrophone to hydrophone in the receiving array. Each interferer in a realization of a set of PM interferences, whether resolvable or nonresolvable, is assumed to be a stationary Gaussian random process. Interferences in the same and in different realizations of the PM interference process are assumed statistically independent. Finally, all interferers are assumed to have identical power spectra. Nonidentical interference spectra can be incorporated into the current approach via so-called ''marked'' Poisson processes (if the spectral mark process is independent of the arrival direction process), but this topic is not pursued here.

Nonhomogeneous Poisson point processes have been used previously in the sonar context. Heitmeyer, Davis, and Yen[2] study the special case of azimuthal directional arrays and use Poisson processes to model shipping anisotropy. Their formulation of the problem differs significantly from the one used in this paper, and they describe their results in

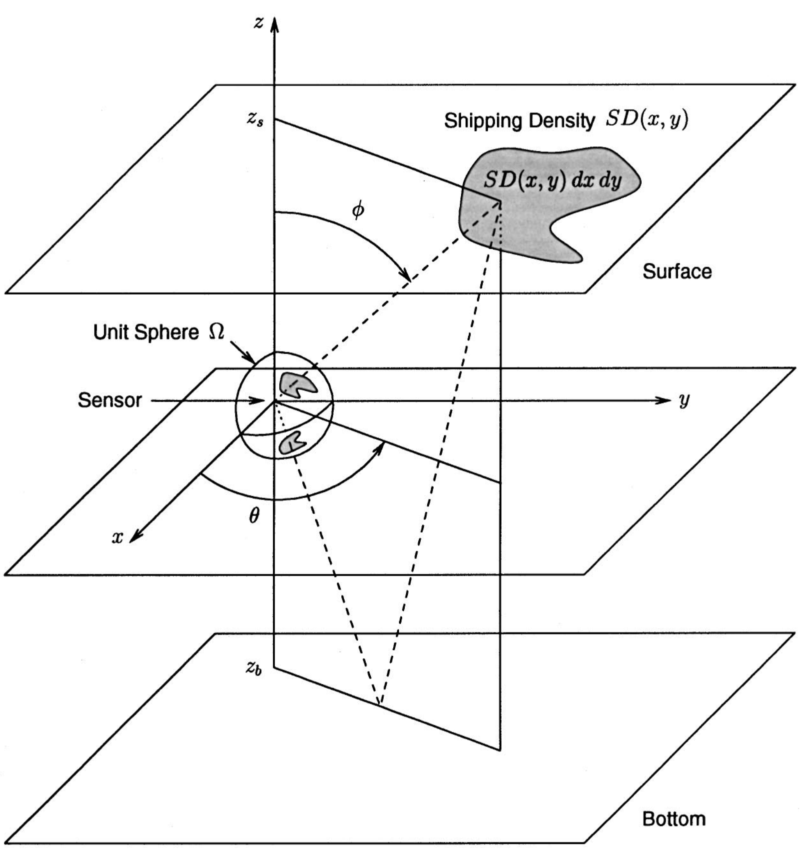

FIG. 2. Acoustic propagation transformation of shipping density $\mathrm{SD}(x,y)$ to interference intensity $\Lambda(\xi)$ on the sphere $\Omega=\{\xi\in R^3:\|\xi\|=1\}$. The case of constant sound speed, direct and bottom bounce paths, and flat bottom is depicted.

terms of a (suitably defined) ‘‘detection opportunity function’’ and ‘‘ship resolution gain.’’ In any event, they do not obtain correction terms to the classical expression for recognition differential of the passive-sonar equation. Liu and Nolte[3] obtain closed-form expressions for the probabilities of detection and false alarm for a single interferer from a fixed, known direction. They cite several papers related to spatial discrimination of array signal processors against interference. Claus, Kadota, and Romain[4] present a method for estimating spatially localized noise embedded in spatially and temporally white ambient noise, and apply it to the narrowband signal detection problem. Their method may be useful for estimating the intensity function of a PM interference field. Van Trees[5] uses nonhomogeneous Poisson processes to model spatially distributed interference (reverberation) in the problem of detecting the (active sonar) return from a slowly fluctuating point target. He states that the spatial Poisson process model is an adequate model of the reverberant environment in many situations.

II. FIXED INTERFERENCES

Deflection coefficients are derived in this section for a fixed number of interferences with specified arrival directions and one signal arrival direction. Let ξ_l denote a unit vector in the look, or steer, direction of an M hydrophone passive array, and let $p_m=(x_m,y_m,z_m)$ denote the locations of the array's hydrophones. The (voltage) beamformer output is the weighted and time-delayed coherent sum of the hydrophone outputs

$$v(t)=\sum_{m=1}^{M} w_m u_m(t-\mu(p_m,\xi_l)), \tag{2.1}$$

where $\{u_m(t)\}$ are the hydrophone outputs, $\{\mu(p_m,\xi_l)\}$ are the time delays corresponding to the wavefront curvature of the signal as it propagates across the array, and $\{w_m\}$ are the array weights. Plane wave propagation is assumed in this paper, so the time-delay function referenced to the origin (which is taken to be the acoustic center of the array) for arbitrary propagation directions ξ and locations p is given by $\mu(p,\xi)=(p\cdot\xi)/c$, where c denotes wave speed across the array. The output of hydrophone m due to a fixed far-field target signal with plane wave arrival direction ξ_s and array hydrophone noise $n_m(t)$ is given by

$$u_m(t)=s(t+\mu(p_m,\xi_s))+n_m(t), \tag{2.2}$$

where $s(t)$ is the output due to signal alone of a hypothetical reference hydrophone located at the origin. Let $V(\omega)$, $S(\omega)$, and $N_m(\omega)$ denote the power spectra of the beamformer output $v(t)$, signal $s(t)$, and hydrophone noises $\{n_m(t)\}$, respectively. The beamformer output spectrum is

$$V(\omega)=S(\omega)|B(\omega;\xi_s,\xi_l)|^2+\sum_{m=1}^{M} N_m(\omega)w_m^2, \tag{2.3}$$

where the beam pattern (or transfer function) of the array is defined for arbitrary plane wave arrival angles ξ by

$$|B(\omega;\xi,\xi_l)|^2 =\left|\sum_{m=1}^{M} w_m \exp[-i\omega(\mu(p_m,\xi)-\mu(p_m,\xi_l))]\right|^2. \tag{2.4}$$

The array weights are normalized so that

$$B(\omega;\xi_l,\xi_l)=\sum_{m=1}^{M} w_m=M. \tag{2.5}$$

The hydrophone noises are now assumed to have identical spectra, so that $N_m(\omega)\equiv N(\omega)$, $m=1,\ldots,M$. Define the (dimensionless) ratio

$$G=\frac{[\sum_{m=1}^{M}w_m]^2}{\sum_{m=1}^{M}w_m^2}=\frac{M^2}{w'w}, \tag{2.6}$$

where $w=(w_1,\ldots,w_M)'$ denotes the array weight vector (and primes denote matrix/vector transpose). G is the array gain for a plane wave and uncorrelated hydrophone noise. Substituting (2.5) and (2.6) into (2.3) gives the more intuitive form

$$V(\omega)=S(\omega)|B(\omega;\xi_s,\xi_l)|^2+G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2. \tag{2.7}$$

The expression (2.7) is valid for any array weight normalization. In the absence of signal, the beamformer output spectrum is the noise spectrum multiplied by the scalar $G^{-1}|B(\omega;\xi_l,\xi_l)|^2=w'w$.

Fixed interference arises when (point) sources other than the target lie in the far field and contribute coherently to the beamformer output. Let K denote the number of interfering sources, and let $\{\xi_k\}$ denote unit vectors in their arrival directions. For $k=1,\ldots,K$, let $\iota_k(t)$ denote the output of a hypothetical reference hydrophone located at the origin when only interferer k is present. Output $\iota_k(t)$ is modeled as a stationary Gaussian process. Interfering sources are assumed independent of each other; that is, outputs $\iota_k(t)$ and $\iota_j(t)$ are statistically independent for $k\neq j$. Signal, interference, and noise are independent, so the beamformer output power spectrum in the presence of K interference sources is, by linear superposition,

$$V(\omega)=S(\omega)|B(\omega;\xi_s,\xi_l)|^2+G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2 +\sum_{k=1}^{K} I_k(\omega)|B(\omega;\xi_k,\xi_l)|^2, \tag{2.8}$$

where $I_k(\omega)$ is the spectrum of the interfering source arriving from direction ξ_k.

Interferences are stationary Gaussian random processes by assumption; thus, detector performance is governed by the deflection coefficient d, where d is the difference in the mean output level of $z(t)$ under the two hypotheses

$$\{\mathbf{H}_0: \text{interference and noise present}\},$$

$$\{\mathbf{H}_1: \text{signal, interference, and noise present}\},$$

divided by the $\mathbf{H}_0$ standard deviation. (Some authors use the detection index, defined as d^2.) The mean of $z(t)$ under $\mathbf{H}_1$ is obtained from (2.8), and its mean under $\mathbf{H}_0$ is obtained from (2.8) by setting $S(\omega)=0$; hence, the difference in the means of $z(t)$ is

$$\Delta=\int_{-\infty}^{\infty} S(\omega)|B(\omega;\xi_s,\xi_l)|^2 d\omega. \tag{2.9}$$

Assuming low signal to aggregate interference plus noise ratio, the variance of $z(t)$ is approximately the same under both hypotheses. The time constant T of the postdetection low-pass filter is assumed large compared to the correlation times of the interference and noise, so that the approximation (B2) can be used. The low-pass filter used here has weight $1/T$ for $|t|\leqslant T/2$, and weight zero otherwise.

The deflection coefficient for the (nonadaptive) conventional detector is obtained using a predetection filter whose transfer function is constant. In this case, from Eq. (B7) in Appendix B,

$$d^K_{\text{conv}}(\xi_l)=\frac{\frac{1}{2}\sqrt{T/\pi}\int_{-\infty}^{\infty}S(\omega)|B(\omega;\xi_s,\xi_l)|^2 d\omega}{\{\int_{-\infty}^{\infty}[G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2+\Sigma_{k=1}^K I_k(\omega)|B(\omega;\xi_k,\xi_l)|^2]^2 d\omega\}^{1/2}}. \tag{2.10}$$

If the noise spectrum is strictly positive (i.e., nonzero) for all ω, and if there is no multipath and the array is steered on target, so that $\xi_s=\xi_l$, then $d^K_{\text{conv}}(\xi_l)$ takes the more intuitive form

$$d^K_{\text{conv}}(\xi_l)=\sqrt{\pi T}GS\left\{\int_{-\infty}^{\infty}N^2(\omega)\left[1+\sum_{k=1}^{K}\frac{I_k(\omega)}{N(\omega)}\frac{|B(\omega;\xi_k,\xi_l)|^2}{w'w}\right]^2 d\omega\right\}^{-1/2}, \tag{2.11}$$

where $S=1/2\pi\int_{-\infty}^{\infty}S(\omega)d\omega$ denotes signal power.

The Eckart filter maximizes the deflection coefficient by choice of the optimal predetection filter. Using (2.8) and substituting into expression (B8) in Appendix B gives the Eckart filter as

$$|H^K_{\text{Eckart}}(\omega,\xi_l)|^2=\frac{S(\omega)|B(\omega;\xi_s,\xi_l)|^2}{[G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2+\Sigma_{k=1}^K I_k(\omega)|B(\omega;\xi_k,\xi_l)|^2]^2}. \tag{2.12}$$

Using the array weight normalization (2.5), the deflection coefficient (B9) is

$$d^K_{\text{Eckart}}(\xi_l)=\frac{1}{2}\left\{\frac{T}{\pi}\int_{\Psi}\frac{S^2(\omega)|B(\omega;\xi_s,\xi_l)|^4}{[G^{-1}M^2N(\omega)+\Sigma_{k=1}^K I_k(\omega)|B(\omega;\xi_k,\xi_l)|^2]^2}d\omega\right\}^{1/2}, \tag{2.13}$$

where Ψ is the (radian) frequency band of interest in the application for which the processor is being designed. If the noise spectrum is strictly positive on Ψ, and the array is steered on target, $\xi_s=\xi_l$, the expressions (2.12) and (2.13) take the more insightful forms

$$|H^K_{\text{Eckart}}(\omega,\xi_l)|^2=\frac{S(\omega)}{N^2(\omega)}\times\left[1+\sum_{k=1}^{K}\frac{I_k(\omega)}{N(\omega)}\frac{|B(\omega;\xi_k,\xi_l)|^2}{w'w}\right]^{-2}, \tag{2.14}$$

and

$$d^K_{\text{Eckart}}(\xi_l)=\frac{G}{2}\left\{\frac{T}{\pi}\int_{\Psi}\left[\frac{S(\omega)}{N(\omega)}\right]^2\times\left[1+\sum_{k=1}^{K}\frac{I_k(\omega)}{N(\omega)}\frac{|B(\omega;\xi_k,\xi_l)|^2}{w'w}\right]^{-2}d\omega\right\}^{1/2}. \tag{2.15}$$

When interference is absent, these results reduce to the corresponding expressions in Appendix B when the array gain $G=1$, that is, when the array comprises a single hydrophone.

III. TEMPORALLY NONRESOLVABLE INTERFERENCES

Temporally nonresolvable interference in the beamformer output arises in complex environments in which both the number and location of the interferences change rapidly. If the time duration of an interferer is sufficiently short and the occurrence rate is sufficiently high when compared to the averaging time T of the postdetection filter, then the detector output $z(t)$ is essentially stationary. The post- and predetection filters are linear and time invariant, so the beamformer output $v(t)$ is also essentially stationary. Consequently, the conventional and Eckart filters for nonresolvable interference can be approximated by this stationary limiting case. It will be shown that nonresolvable PM interference is closely related to classical (planewave) nonisotropic noise.

Assuming nonresolvable interference, the power spectrum of the beamformer output $v(t)$ is evaluated as a double expectation. The first expectation yields the beamformer output power spectrum (2.8), conditioned on a fixed number and location of interferences. The second expectation is the ensemble average of (2.8) with respect to the number K and the directions $\{\xi_1,\xi_2,\ldots,\xi_K\}$ of the nonresolvable interferences.

The PM interference model is defined on the set of all possible arrival directions, namely the unit sphere, denoted by Ω. Let $\Lambda(\xi)\geqslant 0$ denote the occurrence intensity of the

Poisson process in the direction ξ. The subset of Ω over which the intensity attains its maximum value is the set of directions from which interference is most likely to arrive. Similarly, the subset of Ω for which $\Lambda(\xi)=0$ gives the set of directions from which interference is impossible (with probability one). The magnitude of $\Lambda(\xi)$ determines the number of interferers from direction ξ to expect over time, that is, over the statistical ensemble defining the Poisson process. Further discussion of Poisson processes is given in Appendix A.

The expected value of $V(\omega)$ with respect to the PM interference process with intensity $\Lambda(\xi)$ is the PM power spectrum of the beamformer output, denoted by $\overline{V_P}(\omega)$. Denote the interference-weighted expected beam-response pattern by the surface integral

$$\overline{B_\Lambda}(\omega,\xi_l)=\int_\Omega |B(\omega;\xi,\xi_l)|^2\Lambda(\xi)\,d\xi, \tag{3.1}$$

where the surface area element on Ω is $d\xi=\sin\phi\, d\phi\, d\theta$. It is now assumed that the interferers have identical power spectra, $I_k(\omega)=I(\omega)$ for all k. Using Eq. (2.8) and the identity (A6) of Appendix A, the PM power spectrum is easily shown to be given by

$$\begin{aligned}\overline{V_P}(\omega)&=E_\Lambda[V(\omega)]\\&=S(\omega)|B(\omega;\xi_s,\xi_l)|^2+G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2\\&\quad+I(\omega)\overline{B_\Lambda}(\omega,\xi_l).\end{aligned} \tag{3.2}$$

Application-specific expressions for $\overline{V_P}(\omega)$ are obtained by evaluating $\overline{B_\Lambda}(\omega,\xi_l)$ for particular interference intensity functions $\Lambda(\xi)$.

The expectation $\overline{V_P}(\omega)$ for the Poisson interference model differs from nonisotropic (plane wave) ambient noise models only in the interpretation that is made of the intensity function $\Lambda(\xi)$. Ambient noise in the vicinity of the array is classically modeled as the linear superposition of independent plane waves from every possible arrival direction ξ. In contrast to the PM interference model, the plane waves comprising nonisotropic noise are always present. Noise intensity varies with direction ξ. Let $\Lambda(\xi)d\xi$ be the plane wave intensity in direction ξ and differential element $d\xi$. The power spectrum $I(\omega,\xi)\equiv I(\omega)$ of the nonisotropic noise field is spatially homogeneous. Thus, the component of the beamformer output power spectrum due to nonisotropic ambient noise is the integral

$$I(\omega)\int_\Omega |B(\omega;\xi,\xi_l)|^2\Lambda(\xi)\,d\xi, \tag{3.3}$$

which is identical to the third term on the right-hand side of (3.2). At the beamformer output, therefore, PM interference is equivalent to a nonisotropic ambient noise field with intensity $\Lambda(\xi)$. In other words, sufficiently rapid intermittent fluctuations of PM distributed interference is indistinguishable from nonisotropic ambient noise.

If a temporally nonresolvable PM signal model is employed to model signal multipath effects, then it is also necessary to take the expectation of (2.8) with respect to the PM signal process. Let $\Gamma(\xi)$ denote the Poisson intensity of signal arrival direction. Signal and interference PM processes are independent; hence, paralleling the above discussion for interference gives the PM spectrum with respect to signal and interference processes as

$$\begin{aligned}\bar{\bar{V}}_P(\omega)&=E_\Gamma[\overline{V_P}(\omega)]\\&=S(\omega)\overline{B_\Gamma}(\omega,\xi_l)+G^{-1}N(\omega)|B(\omega;\xi_l,\xi_l)|^2\\&\quad+I(\omega)\overline{B_\Lambda}(\omega,\xi_l),\end{aligned} \tag{3.4}$$

where $\overline{B_\Gamma}(\omega,\xi_l)$ is given by (3.1) with Γ replacing Λ. If the signal arrives from a single direction, say ξ_{s_0}, then $\Gamma(\xi)\equiv\gamma_0\delta(\xi-\xi_{s_0})$, where $\delta(\cdot)$ denotes the Dirac delta function; the result (3.4) reverts to (3.2) exactly if the intensity $\gamma_0=1$.

From Eq. (3.4), the difference in means of $z(t)$ for PM signal and interference processes under hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ is

$$\Delta=\int_{-\infty}^{\infty} S(\omega)\overline{B_\Gamma}(\omega,\xi_l)\,d\omega. \tag{3.5}$$

The deflection coefficients are obtained from (3.5) and the variance of the PM spectrum (3.4) under a low SNR assumption. From Eq. (B4) in Appendix B and the normalization $B(\omega;\xi_l,\xi_l)=M$, it follows that

$$\begin{aligned}\mathrm{Var}[\bar{\bar{V}}_P(\omega)]&=\frac{1}{\pi T}\int_{-\infty}^{\infty}|H_0(\omega)|^4[G^{-1}M^2N(\omega)\\&\quad+I(\omega)\overline{B_\Lambda}(\omega,\xi_l)]^2\,d\omega,\end{aligned} \tag{3.6}$$

assuming the correlation times of noise and interferences are small compared to T, the averaging time of the postdetection filter. For short-duration interferers, this assumption is valid. The predetection filter is omitted from the conventional detector, so that $x(t)\equiv v(t)$ in Fig. 1, that is, $|H_0(\omega)|\equiv 1$. From (B7) of Appendix B

$$d_{\mathrm{conv}}^{\mathrm{nonres}}(\xi_l)=\frac{\frac{1}{2}\sqrt{T/\pi}\int_{-\infty}^{\infty}S(\omega)\overline{B_\Gamma}(\omega,\xi_l)\,d\omega}{\{\int_{-\infty}^{\infty}[G^{-1}M^2N(\omega)+I(\omega)\overline{B_\Lambda}(\omega,\xi_l)]^2\,d\omega\}^{1/2}}. \tag{3.7}$$

If there is no signal multipath and $\Gamma(\xi)\equiv\gamma_0\delta(\xi-\xi_{s_0})$ with $\gamma_0=1$, if the array is steered on target, so that $\xi_{s_0}=\xi_l$, and if the noise spectrum is strictly positive, then (3.7) can be written in the more insightful form

$$\begin{aligned}d_{\mathrm{conv}}^{\mathrm{nonres}}(\xi_l)&=\sqrt{\pi T}GS\left\{\int_{-\infty}^{\infty}N^2(\omega)\right.\\&\quad\left.\times\left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B_\Lambda}(\omega,\xi_l)}{w'w}\right]^2d\omega\right\}^{-1/2}.\end{aligned} \tag{3.8}$$

This result is what would be obtained from Eq. (2.11) by setting $I_k(\omega)\equiv I(\omega)$ for all k and replacing the finite sum by its expectation with respect to the PM interference model.

If the noise spectrum is strictly positive, the optimal Eckart filter is

$$|H^{\text{nonres}}_{\text{Eckart}}(\omega,\xi_l)|^2 = \frac{S(\omega)\overline{B}_\Gamma(\omega,\xi_l)}{N^2(\omega)} \times \left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B}_\Lambda(\omega,\xi_l)}{w'w}\right]^{-2}, \tag{3.9}$$

and the deflection coefficient is

$$d^{\text{nonres}}_{\text{Eckart}}(\xi_l) = \frac{G}{2}\left\{\frac{T}{\pi}\int_\Psi\left[\frac{S(\omega)\overline{B}_\Gamma(\omega,\xi_l)}{N(\omega)}\right]^2 \times\left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B}_\Lambda(\omega,\xi_l)}{w'w}\right]^{-2} d\omega\right\}^{1/2}. \tag{3.10}$$

The results (3.9) and (3.10) are the same as would be obtained from (2.14) and (2.15), respectively, by replacing finite sums with their analogous Poisson expectations.

IV. TEMPORALLY RESOLVABLE INTERFERENCES

Temporally resolvable interference occurs when interferences in the far field of the array are of long duration compared to the averaging time T of the postdetection filter. The start and end times of such interferers can be estimated to within an error depending on T. These interferers are, essentially, models of distant ship traffic. In practice, the arrival directions of these interferers can be estimated and their effect on detection analyzed using the fixed interference model of Sec. II. This approach is useful for specific interference distributions, but it does not give insight into detection capabilities when the specific traffic pattern is unknown. In this section, ensemble averages (expectations) of mean levels and standard deviations are evaluated explicitly under the assumption that the number and location of the interferers follow the PM interference statistical model. The resulting expression for the deflection coefficient d depends on the PM interference intensity function, but it is independent of specific interference patterns.

Because only the signal contribution remains after differencing the conditional mean of the output $z(t)$ under hypotheses $\mathbf{H}_1$ and $\mathbf{H}_0$, the deflection coefficient is easily computed from the variance of $z(t)$ under $\mathbf{H}_0$. Conditioned on a fixed number and location of interferences, the variance of $z(t)$, denoted

$$\text{Var}[z] \equiv E[(z-\overline{z})^2|\text{fixed interference}], \tag{4.1}$$

is stationary; that is, $\text{Var}[z]$ is constant because the ensemble used to define $\text{Var}[z]$ comprises only realizations of $z(t)$ having a fixed interference pattern. However, PM interference is nonstationary, and $\text{Var}[z]$ fluctuates over time. As a result, it is necessary to compute the expectation of $\text{Var}[z]$ with respect to the PM interference model. The expectation $E_\Lambda[\text{Var}[z]]$ is thus a double expectation of the same statistical character as used above for nonresolvable interference.

An explicit expression for the variance (4.1) is obtained by assuming that the correlation time of an interferer is small compared to the averaging time T of the postdetection filter. Thus, in comparison to T, resolvable interferers have long durations and short correlation times. Correlation time is approximately inversely proportional to bandwidth, so the assumption is satisfied in applications in which the interference bandwidth is large compared to $2\pi/T$.

From the small SNR assumption, it follows from (2.8) with $I_k(\omega)=I(\omega)$ for all k and from (B4) of Appendix B that

$$\text{Var}[z] = \frac{1}{\pi T}\int_{-\infty}^{\infty}|H_0(\omega)|^4\left[G^{-1}M^2N(\omega) + I(\omega)\sum_{k=1}^{K}|B(\omega;\xi_k,\xi_l)|^2\right]^2 d\omega,$$

where the normalization $B(\omega;\xi_l,\xi_l)=M$ has been used. Linearity of the expectation operator gives the expectation of $\text{Var}[z]$ with respect to the PM interference model as

$$E_\Lambda[\text{Var}[z]] = \frac{1}{\pi T}\int_{-\infty}^{\infty}|H_0(\omega)|^4 E_\Lambda\left[\left[\frac{M^2}{G}N(\omega) + I(\omega)\sum_{k=1}^{K}|B(\omega;\xi_k,\xi_l)|^2\right]^2\right]d\omega. \tag{4.2}$$

Interchanging the integral and the expectation operator is justified here because the integral is absolutely convergent. From (A9) of Appendix A, the required expectation in the integrand of (4.2) is

$$\left[G^{-1}M^2N(\omega)+I(\omega)\int_\Omega|B(\omega;\xi,\xi_l)|^2\Lambda(\xi)d\xi\right]^2 + I^2(\omega)\int_\Omega|B(\omega;\xi,\xi_l)|^4\Lambda(\xi)d\xi.$$

The first term was encountered earlier with nonresolvable interference; the second term is contributed by the nonstationary character of resolvable PM interference. The predetection filter is omitted from the conventional detector, so that $|H_0(\omega)|=1$; hence, from (B7) of Appendix B, the deflection coefficient for the conventional processor in resolvable PM interference is

$$d^{\text{res}}_{\text{conv}}(\xi_l) = \frac{\frac{1}{2}\sqrt{\frac{T}{\pi}}\int_{-\infty}^{\infty}S(\omega)\overline{B}_\Gamma(\omega,\xi_l)d\omega}{\tilde{\sigma}_{\text{res}}}, \tag{4.3}$$

where

$$(\tilde{\sigma}_{\text{res}})^2 = \int_{-\infty}^{\infty}\left[G^{-1}M^2N(\omega) + I(\omega)\int_\Omega|B(\omega;\xi,\xi_l)|^2\Lambda(\xi)d\xi\right]^2 d\omega + \int_{-\infty}^{\infty}\int_\Omega I^2(\omega)|B(\omega;\xi,\xi_l)|^4\Lambda(\xi)d\xi\, d\omega. \tag{4.4}$$

If there is no signal multipath and $\Gamma(\xi)=\delta(\xi-\xi_{s_0})$, if the array is steered on target so that $\xi_l=\xi_{s_0}$, and if the noise spectrum is positive, then

$$d_{\rm conv}^{\rm res}(\xi_l)=\sqrt{\pi T}GS\left\{\int_{-\infty}^{\infty}N^2(\omega)\left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B}_\Lambda(\omega,\xi_l)}{w'w}\right]^2 d\omega\right.$$
$$\left.+\int_{-\infty}^{\infty}I^2(\omega)\left[\frac{\int_\Omega|B(\omega;\xi,\xi_l)|^4\Lambda(\xi)d\xi}{(w'w)^2}\right]d\omega\right\}^{-1/2}. \tag{4.5}$$

The deflection coefficient reduces to the interference-free case if $I(\omega)=0$.

The Eckart filter is given by

$$|H_{\rm Eckart}^{\rm res}(\omega,\xi_l)|^2=\frac{S(\omega)\overline{B}_\Gamma(w,\xi_l)}{N^2(\omega)}\left\{\left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B}_\Lambda(\omega,\xi_l)}{w'w}\right]^2\right.$$
$$\left.+\left[\frac{I(\omega)}{N(\omega)}\right]^2\frac{\int_\Omega|B(\omega;\xi,\xi_l)|^4\Lambda(\xi)d\xi}{w'w}\right\}^{-1}, \tag{4.6}$$

and the corresponding deflection coefficient is

$$d_{\rm Eckart}^{\rm res}(\xi_l)=\frac{G}{2}\left\{\frac{T}{\pi}\int_\Psi\left[\frac{S(\omega)\overline{B}_\Gamma(w,\xi_l)}{N(\omega)}\right]^2\right.$$
$$\times\left\{\left[1+\frac{I(\omega)}{N(\omega)}\frac{\overline{B}_\Lambda(\omega,\xi_l)}{w'w}\right]^2\right.$$
$$\left.\left.+\left[\frac{I(\omega)}{N(\omega)}\right]^2\frac{\int_\Omega|B(\omega;\xi,\xi_l)|^4\Lambda(\xi)d\xi}{w'w}\right\}^{-1}d\omega\right\}^{1/2}. \tag{4.7}$$

These results reduce to the interference-free case if $I(\omega)=0$.

V. RECOGNITION DIFFERENTIALS

Classical expressions for the recognition differential ($N_{\rm RD}$) are obtained by imposing simple spectral models on both signal and noise and then taking the logarithm of the deflection coefficient for the conventional processor in the interference-free case (Cox[6]). In this section, these spectral models are supplemented with an equally simple interference spectral model. Interference correction (IC) factors for the classical $N_{\rm RD}$ expression are derived via the appropriate deflection coefficient expression for fixed interferences and for both temporally resolvable and nonresolvable interferences. (Alternatively, the classical $N_{\rm RD}$ expression can be retained if the IC term is treated as a noise term and added to the sonar equation.)

Creese[7] uses the results presented in this section to evaluate $N_{\rm RD}$ expressions numerically for a uniformly shaded line array. He considers several ocean environments with different shipping densities and sea states. Multiple types of interference spectra are modeled by superposition of several PM interference models, one for each spectral type. For further details, the interested reader is referred to Creese.[7]

The classical $N_{\rm RD}$ analysis is straightforward. If there is no interference, $I(\omega)\equiv 0$; if the noise spectrum is flat and equals N_0 (per Hertz) over a frequency band of (radian) bandwidth W, and if there are no signal multipath effects, then from (2.11)

$$d_{\rm conv}^K(\xi_l)|_{K=0}=G\sqrt{\frac{TW_0}{2\pi}}\frac{S}{N}, \tag{5.1}$$

where

$$N=\frac{1}{2\pi}\int_{-\infty}^{\infty}N(\omega)d\omega=\frac{WN_0}{2\pi} \tag{5.2}$$

is the noise power and $W_0=W/2$ denotes the (radian) bandwidth of the one-sided noise spectrum. In this case, the deflection coefficient for the conventional processor is directly proportional to the array gain G, the square root of the time-bandwidth product TW_0, and the signal-to-noise ratio. Let $d_{\rm thresh}$ denote the detection threshold determining the $(P_D,P_{\rm FA})$ operating point on the detector's ROC curve. Converting to decibels (dB) by taking $10\log$ of (5.1) and solving for signal gives

$$N_{\rm RD}=SL-NL+AG, \tag{5.3}$$

where

$$SL=10\log S,\quad NL=10\log N,\quad AG=10\log G,$$

and

$$N_{\rm RD}=DT-5\log\left(\frac{TW_0}{2\pi}\right), \tag{5.4}$$

where $DT=10\log d_{\rm thresh}$. Equation (5.3) is the classical passive-sonar equation if SL, the received signal level, is set equal to the target source level minus the one-way transmission loss. In practice, background noise level is estimated from beams that are free of signal and interference; however, such procedures are unnecessary in a theoretical analysis if noise and interference models are given.

The classical recognition differential (5.4) does not depend on the look direction ξ_l when the assumptions under which it is derived are met; that is, when there is no interference. When interference is present, the recognition differential is written

$$N_{\rm RD}=DT-5\log\left(\frac{TW_0}{2\pi}\right)+{\rm IC},$$

where the term IC depends on the specific nature of the interference.

For fixed interferences, the deflection coefficient (2.11) holds. Assuming that the noise and interference spectra are flat over the same bandwidth W, and that interferences have identical spectra equal to I_0 (per Hertz), the IC term is

$${\rm IC}^K=5\log\left[\frac{1}{W}\int_W\left[1+\frac{I}{N}\sum_{k=1}^{K}\frac{|B(\omega;\xi_k,\xi_l)|^2}{w'w}\right]^2d\omega\right], \tag{5.5}$$

where $I=1/2\pi\int_{-\infty}^{\infty}I(\omega)d\omega=WI_0/2\pi$ is the interference power. For the case $K=0$, the correction term ${\rm IC}^K\equiv 0$.

The correction term (5.5) simplifies if the integral over the bandwidth W is approximated by the product of W and the integrand evaluated at the center frequency ω_c of the one-sided band W_0. With this approximation, (5.5) becomes

$$\mathrm{IC}^K \cong 10\log\left[1+\frac{I}{N}\sum_{k=1}^{K}\frac{|B(\omega_c;\xi_k,\xi_l)|^2}{w'w}\right]. \tag{5.6}$$

The IC term (in dB) for K-independent interferences is not the linear sum of K individual IC terms, as is clear from both (5.5) and the approximation (5.6).

The derivation of the correction term for nonresolvable PM interference proceeds in essentially the same manner, but with the sum in (5.5) replaced by an integral. Using the deflection (3.8), the IC term takes the form

$$\mathrm{IC}^{\mathrm{nonres}} = 5\log\left[\frac{1}{W}\int_W\left[1+\frac{I}{N}\int_\Omega\frac{|B(\omega;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right]^2 \times d\omega\right]. \tag{5.7}$$

If the interference power $I=0$, the correction term (5.7) is zero. The correction term (5.7) simplifies if the product of W and the integrand evaluated at the center frequency ω_c approximates the integral over ω. Substituting this approximation in (5.7) gives

$$\mathrm{IC}^{\mathrm{nonres}} \cong 10\log\left[1+\frac{I}{N}\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right]. \tag{5.8}$$

Limiting values of (5.8) as $I/N\to 0$ and as $I/N\to\infty$ are of interest. Using the approximation $\log_e(1+x)\cong x$, $x\to 0$, gives the low interference-to-noise limit, in dB, as

$$\mathrm{IC}^{\mathrm{nonres}} \cong 4\left\{\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right\}\frac{I}{N}, \quad \frac{I}{N}\to 0, \tag{5.9}$$

where $4\cong 10\log e$. Thus, (5.9) approaches zero as $I/N\to 0$. The high interference-to-noise limit is

$$\begin{aligned}\mathrm{IC}^{\mathrm{nonres}} &\cong 10\log\left[\frac{I}{N}\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right], \quad \frac{I}{N}\to\infty,\\ &= IL-NL+10\log\left[\frac{\int_\Omega|B(\omega_c;\xi,\xi_l)|^2\Lambda(\xi)d\xi}{w'w}\right],\end{aligned} \tag{5.10}$$

where $IL=10\log I$ is the interference power in dB. The high interference-to-noise limit is obtained merely by noting that the constant 1 in (5.8) is small compared to I/N.

The correction term for resolvable PM interference proceeds from the deflection coefficient given by (4.5). The IC term is

$$\mathrm{IC}^{\mathrm{res}} = 5\log\left[\frac{1}{W}\int_W\left\{\left[1+\frac{I}{N}\int_\Omega\frac{|B(\omega;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right]^2 + \left(\frac{I}{N}\right)^2\int_\Omega\frac{|B(\omega;\xi,\xi_l)|^4}{(w'w)^2}\Lambda(\xi)d\xi\right\}d\omega\right]. \tag{5.11}$$

Using the product of W and the integrand evaluated at the center frequency ω_c to approximate the integral over ω gives

$$\mathrm{IC}^{\mathrm{res}} \cong 5\log\left[\left[1+\frac{I}{N}\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right]^2 + \left(\frac{I}{N}\right)^2\int_\Omega\frac{|B(\omega_c,\xi,\xi_l)|^4}{(w'w)^2}\Lambda(\xi)d\xi\right]. \tag{5.12}$$

The low interference-to-noise limit is the same as (5.9), as is seen from (5.12) by noting that the ratio $(I/N)^2$ is small compared to I/N. The high interference-to-noise limit is

$$\begin{aligned}\mathrm{IC}^{\mathrm{res}} &\cong 5\log\left[\left(\frac{I}{N}\right)^2\left\{\left[\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right]^2 + \int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^4}{(w'w)^2}\Lambda(\xi)d\xi\right\}\right]\\ &= IL-NL+10\log\left[\frac{I}{N}\int_\Omega\frac{|B(\omega_c;\xi,\xi_l)|^2}{w'w}\Lambda(\xi)d\xi\right],\\ &\quad +5\log\left[1+\frac{\int_\Omega|B(\omega_c;\xi,\xi_l)|^4\Lambda(\xi)d\xi}{\{\int_\Omega|B(\omega_c;\xi,\xi_l)|^2\Lambda(\xi)d\xi\}^2}\right].\end{aligned} \tag{5.13}$$

Thus, as $I/N\to\infty$, the IC term for resolvable interference differs from that for nonresolvable interference by the last term in (5.13), which depends on the directional process $\Lambda(\xi)$ but not on the ratio I/N.

VI. SUMMARY AND CONCLUDING REMARKS

A quantitative analysis of the degradation of performance for the conventional and optimal detector for arrays of arbitrary geometry has been presented for fixed interference and for nonresolvable and resolvable PM interference. Rapid, short duration, temporally nonresolvable interference leads to expressions that are essentially the same as those obtained from classical nonisotropic noise models. Long duration, temporally resolvable interference is slowly varying when compared to the averaging time of the postdetection filter, and it leads to novel expressions for the deflection coefficient. Interference correction terms for the conventional detector are derived from the deflection coefficients for all three kinds of interference.

Poisson interference models for specific scenarios with slowly changing, but relatively stable, interference fields can be obtained by extending the work presented here to doubly stochastic Poisson models; that is, to models in which the Poisson interference intensity is a function of time. Resolvable and nonresolvable interferences are limiting cases of this more general temporal–directional PM interference model. The approach requires parametrizing the Poisson intensity functions of signal and interference appropriately and estimating the constituent parameters adaptively over time. The beam level deflection coefficients $\{d(\xi_l)\}$ are thus a collection of coupled (especially at lower frequencies in which interference arrival directions are not spatially resolvable) sufficient statistics of a family of generalized likelihood ratio tests in which the signal, noise, and interference spectra are jointly statistically estimated by maximum likelihood or maximum *a posteriori* multibeam procedures. Generalized likelihood ratio tests are often, but not always, very effective in practice, and their performance is usually difficult to ascertain theoretically. An adaptive approach should improve the performance of the conventional and optimal processors in changing interference environments; however, adaptive multibeam filters and similar interesting topics lie outside the scope of the present paper.

Finally, practical experience suggests that the split-beam (or split-array) cross-correlation processor may sometimes offer better performance than the optimum processor in difficult interference environments. The application of PM interference models to analyze the detection performance of the cross-correlation processor may facilitate comparisons between it and the optimal (full-array) processor discussed in this paper.

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APPENDIX A: NONHOMOGENEOUS POISSON PROCESSES

Poisson point processes are discrete–continuous stochastic processes. The discrete component determines the number of events, or points, that occur in a given realization and the continuous component determines where the points lie in a specified space, or domain, denoted by X. The intensity $\Lambda(x)$ of the process is a non-negative function defined for every point $x\in X$, and it completely determines both the discrete and continuous components of the distribution. If $\Lambda(x)$ is constant over X, then the process is said to be homogeneous; otherwise, it is nonhomogeneous. Simulation of one realization of a Poisson process on a given region $S\subset X$ is a two-step procedure. Step 1 samples the discrete component to determine the number of events that occur. The discrete distribution governing this sample is the Poisson distribution with parameter $\lambda_S=\int_S\Lambda(x)dx$, where dx denotes the appropriate differential of ''volume'' in the space X. Because the intensity is non-negative, λ_S is non-negative also. The discrete Poisson distribution is

$$P[k]=\frac{(\lambda_S)^k}{k!}\exp[-\lambda_S],\quad k=0,1,2,\ldots. \tag{A1}$$

Suppose, then, that the realization of the discrete Poisson distribution with parameter λ_S is K; that is, K is the number of points that must now be sampled from the set S. Step 2 of the simulation samples these K points as independent realizations of a random variable whose probability density function is

$$p_S(x)=\frac{\Lambda(x)}{\int_S\Lambda(\xi)d\xi},\quad x\in S. \tag{A2}$$

This completes the simulation of one realization of the Poisson process. The two-step procedure depends on the choice of the subset S via the parameter λ_S and the density $p_S(x)$. If λ_S is zero for some set S, then with probability one the outcome of *step* 1 is $K=0$, so that no points need be sampled in *step* 2; thus, the possible division by zero in (A2) is avoided. The sample points in *step* 2 are independent of the number K of sample points drawn, so that for all $K\geqslant 1$

$$p_S(x_1,\ldots,x_K|K)=p_S(x_1,\ldots,x_K)=\prod_{k=1}^{K}p_S(x_k). \tag{A3}$$

This important result is used below in the definition of the expectation operator.

The domain X over which a Poisson process can be defined is as general as the concept of integration; that is, it can be defined on any domain X for which $\int_S\Lambda(\xi)d\xi$ is defined for any (measurable) region $S\subset X$. The process is temporal if X is a time interval, say T, spatial if $X=R^p$ for integers $p\geqslant 1$, spatial–temporal if $X=R^k\times T$, directional if $X=\Omega\equiv\{x\in R^3:\|x\|=1\}$, and directional–temporal if $X=\Omega\times T$. It is an azimuthal process if X is the unit circle Ω in R^2. The details change with the space X, but Eqs. (A1) and (A2) always hold.

The expectation operator on the region $S\subset X$ is defined for real or complex valued functions $\mathcal{F}$ as

$$\begin{aligned}E_\Lambda[\mathcal{F}]&=\sum_{K=1}^{\infty}P[K]\int\cdots\int_{S^K}\mathcal{F}(x_1,\ldots,x_K)\\&\quad\times p_S(x_1,\ldots,x_K|K)dx_1,\ldots,dx_K\\&=\sum_{K=1}^{\infty}P[K]\int\cdots\int_{S^K}\mathcal{F}(x_1,\ldots,x_K)\\&\quad\times\prod_{k=1}^{K}p(x_k)dx_1,\ldots,dx_K,\end{aligned} \tag{A4}$$

where the integral is over the K-fold Cartesian product $S\times\cdots\times S\equiv S^K\subset X^K$, the probability $P[K]$ is given by (A1), and where the conditional independence of sample points (A3) is substituted in the second expression. It is evident from (A4) that the function $\mathcal{F}$ must be defined for any number of arguments, so $\mathcal{F}$ cannot be a traditional function having a fixed number of arguments; in other words, $\mathcal{F}$ must be defined for every possible outcome of the Poisson process. The kind of function most relevant to this paper is, for $K>0$,

$$\mathcal{F}=\mathcal{F}(x_1,\ldots,x_K)=\sum_{k=1}^{K}f(x_k), \tag{A5}$$

where $f(x)$ is a traditional function having a single argument. For $K=0$, $\mathcal{F}$ is defined to be 0. The expected value of $\mathcal{F}$ is

$$E_\Lambda[\mathcal{F}]\equiv E_\Lambda\left[\sum_{k=1}^{K}f(x_k)\right]=\int_S f(x)\Lambda(x)dx. \tag{A6}$$

Intuitively, this expectation weights terms in the sum (A5) by the intensity with which those terms can occur. If $f(x)=1$, then (A6) is λ_S, and if $f(x)=\exp(i\omega\cdot x)$, $\omega\in X\equiv R^p$ for some $p\geqslant 1$, then (A6) is the characteristic function of $\Lambda(x)$ restricted to the region S.

The expected value of the product of two functions of the form (A5) is

$$E_\Lambda\left[\sum_{k=1}^{K} f(x_k)\sum_{k=1}^{K} g(x_k)\right]=\int_S f(x)g(x)\Lambda(x)dx + \int_S f(x)\Lambda(x)dx\int_S g(x)\Lambda(x)dx. \tag{A7}$$

Setting $f\equiv g^*$ gives the important result

$$E_\Lambda\left[\left|\sum_{k=1}^{K} f(x_k)\right|^2\right]=\int_S |f(x)|^2\Lambda(x)dx + \left|\int_S f(x)\Lambda(x)dx\right|^2. \tag{A8}$$

The proofs of (A6) and (A7) given by Hero[8] hold for intensities $\Lambda(x)$ such that $\int_S \Lambda(x)dx<\infty$, and for real-valued functions f and g such that

$$\int_S |f(x)|^2\Lambda(x)dx<\infty \quad \text{and} \quad \int_S |g(x)|^2\Lambda(x)dx<\infty.$$

The proof for complex valued functions follows easily by separating them into real and imaginary parts. Finally, if f_0 is an arbitrary constant, then

$$E_\Lambda\left[\left|f_0+\sum_{k=1}^{K} f(x_k)\right|^2\right]=\left|f_0+\int_S f(x)\Lambda(x)dx\right|^2 + \int_S |f(x)|^2\Lambda(x)dx, \tag{A9}$$

as is seen by expanding the squared magnitude in (A9) and substituting the results (A6) and (A8).

For further discussion of Poisson processes, the alternative tradition in signal processing, see Snyder and Miller[9] and Kingman.[10]

APPENDIX B: DETECTION OF GAUSSIAN SIGNALS IN ADDITIVE NOISE

The deflection coefficient for a stationary Gaussian signal in additive independent stationary Gaussian noise is reviewed in this Appendix, as is the optimal predetection, or Eckart, filter. The spectrum of the Gaussian signal that is least detectable by the Eckart processor is also discussed. Robust versions of the Eckart filter that are insensitive to variations in signal and noise spectra are discussed by Al-Husseini and Kassam.[11]

The processing stream is limited to the portion of Fig. 1 from input $v(t)$ to output $z(t)$, inclusive. Predetection and postdetection filters are assumed linear and time invariant. The input $v(t)$ is assumed real valued, zero mean, Gaussian, and stationary, so $x(t)$ is also real valued, zero mean, Gaussian, and stationary. Due to the nonlinear square-law device, $y(t)$ and $z(t)$ are real valued and stationary, but neither zero mean nor Gaussian. (If the input process is complex valued instead of real valued, as assumed in this Appendix, the Eckart filter and corresponding deflection coefficient differ from the expressions given below by a constant.) A complete but standard derivation of the deflection coefficient can be found in Streit.[12]

Assuming small SNR, the output variance under the hypotheses

$$\{\mathbf{H}_0: \text{noise only is present}\},$$

$$\{\mathbf{H}_1: \text{signal plus noise is present}\},$$

are approximately equal. The difference of the means of the output $z(t)$ under these hypotheses is divided by the standard deviation under the null hypothesis to obtain the so-called deflection coefficient, denoted by d. The ROC curves of the detector are completely parametrized by d under Gaussian signal and noise statistical assumptions.

Let $h_1(\tau)$ denote the impulse response function of the postdetection filter normalized so that

$$\int_{-\infty}^{\infty} h_1(\tau)d\tau=1. \tag{B1}$$

The simplest low-pass postdetection filter satisfying the normalization (B1) is $h_1(t)=1/T$ for $|t|\leqslant T/2$ and $h_1(t)=0$ for $|t|\geqslant T/2$. The variance of the output $z(t)$ is given by (see, for example, Thomas[13] for a careful discussion)

$$\mathrm{Var}[z]=\frac{2}{T}\int_{-T}^{T} R_x^2(\tau)\left(1-\frac{|\tau|}{T}\right)d\tau \cong \frac{1}{\pi T}\int_{-\infty}^{\infty} S_x^2(\omega)d\omega, \tag{B2}$$

where $S_x(\omega)$ is the power spectrum of $x(t)$. The approximation (B2) is reasonable if T is large compared to the correlation time. Denoting the impulse response and transfer functions of the predetection filter by $h_0(\tau)$ and $H_0(\omega)$, respectively, gives

$$S_x(\omega)=|H_0(\omega)|^2 S_v(\omega). \tag{B3}$$

Substituting (B3) into (B2) gives

$$\mathrm{Var}[z]=\frac{1}{\pi T}\int_{-\infty}^{\infty} |H_0(\omega)|^4 S_v^2(\omega)d\omega. \tag{B4}$$

The difference in the means of $z(t)$ under hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ is computed from the power spectra of the input $v(t)$ under hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$, denoted by $S_v^N(\omega)$ and $S_v^{S+N}(\omega)$, respectively. Signal and noise are statistically independent, so their power spectra add; that is, $S_v^{S+N}(\omega)=S_v^S(\omega)+S_v^N(\omega)$, where $S_v^S(\omega)$ denotes the spectrum of the signal. The predetection filter is linear and time invariant, so the difference in the mean of the output $z(t)$ under hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ is

$$\Delta\equiv E[z|H_1]-E[z|H_0]=\frac{1}{2\pi}\int_{-\infty}^{\infty} |H_0(\omega)|^2 S_v^S(\omega)d\omega. \tag{B5}$$

The deflection coefficient for Gaussian random variables with identical variances is defined by $d=\Delta/\sigma$, where σ denotes the approximately equal standard deviations of $z(t)$ under $\mathbf{H}_0$ and $\mathbf{H}_1$. Substituting (B5) and (B4) gives

$$d=\frac{\frac{1}{2}\sqrt{\frac{T}{\pi}}\int_{-\infty}^{\infty}|H_0(\omega)|^2 S_v^S(\omega)\,d\omega}{[\int_{-\infty}^{\infty}|H_0(\omega)|^4[S_v^N(\omega)]^2 d\omega]^{1/2}}. \quad \text{(B6)}$$

If the signal spectrum is unknown, the predetection filter is sometimes omitted, so that

$$d=\frac{\frac{1}{2}\sqrt{\frac{T}{\pi}}\int_{-\infty}^{\infty} S_v^S(\omega)\,d\omega}{[\int_{-\infty}^{\infty}[S_v^N(\omega)]^2 d\omega]^{1/2}}, \quad \text{(B7)}$$

which is the expression used in the main body of the paper for the conventional processor.

The deflection (B6) can be maximized by a particular choice of $H_0(\omega)$ in any specified frequency band, say Ψ, for which the noise spectrum is bounded away from zero; that is, $S_v^N(\omega)\geqslant\varepsilon>0$ for $\omega\in\Psi$. The maximizing filter is called the Eckart filter, after its discoverer,[14] and it is denoted here by $H_{\text{Eckart}}(\omega)$. Straightforward application of the Cauchy–Schwarz inequality (see, e.g., Burdic[15]) gives

$$|H_{\text{Eckart}}(\omega)|^2=\frac{S_v^S(\omega)}{[S_v^N(\omega)]^2}. \quad \text{(B8)}$$

Substituting (B8) into (B6) gives

$$d_{\text{Eckart}}=\frac{1}{2}\left[\frac{T}{\pi}\int_\Psi\left[\frac{S_v^S(\omega)}{S_v^N(\omega)}\right]^2 d\omega\right]^{1/2}. \quad \text{(B9)}$$

Given the noise spectrum $S_v^N(\omega)$ and a specified signal power, say S_{power}, the least detectable signal $S_{\text{opt}}(\omega)$ is the signal for which the deflection coefficient d_{Eckart} is a minimum. Let $\alpha\geqslant 0$ denote the SNR. Then, $S_{\text{opt}}(\omega)$ is the solution of the following constrained optimization problem:

$$\min_{S(\omega)}\int_\Psi\left[\frac{S(\omega)}{S_v^N(\omega)}\right]^2 d\omega, \quad \text{(B10)}$$

subject to the SNR constraint

$$\frac{S_{\text{power}}}{\int_\Psi S_v^N(\omega)\,d\omega}=\frac{\int_\Psi S(\omega)\,d\omega}{\int_\Psi S_v^N(\omega)\,d\omega}=\alpha. \quad \text{(B11)}$$

Applying the calculus of variations to the Lagrangian function, it is straightforward to show (see Streit[12] for a detailed discussion) that the minimizing signal is

$$S_{\min}(\omega)=S_{\text{power}}\frac{[S_v^N(\omega)]^2}{\int_\Psi[S_v^N(\omega)]^2 d\omega}. \quad \text{(B12)}$$

In other words, the spectrum of the least detectable signal is directly proportional to the normalized square of the noise spectrum, and the proportionality constant is the specified signal power.

If noise power is finite and its spectrum is continuous and bounded on the band Ψ, then the deflection coefficient d_{Eckart} can be made arbitrarily large for any specified signal power $S_{\text{power}}>0$. For example, if the signal spectrum is

$$S_\epsilon(\omega)=\begin{cases}\epsilon^{-1}S_{\text{power}}, & a\leqslant\omega\leqslant a+\epsilon,\\ 0, & \text{otherwise,}\end{cases}$$

where a and $\epsilon>0$ are such that the interval $[a,a+\epsilon]$ is a subset of Ψ, then it is straightforward to show that $\int_\Psi S_\epsilon(\omega)\,d\omega=S_{\text{power}}$ and

$$\int_\Psi\left[\frac{S_\epsilon(\omega)}{S_v^N(\omega)}\right]^2 d\omega=\frac{1}{\epsilon}\frac{S_{\text{power}}^2}{[S_v^N(\omega_\epsilon)]^2}\geqslant\frac{1}{\epsilon}\left[\frac{S_{\text{power}}}{\max_{\omega\in\Psi}S_v^N(\omega)}\right]^2. \quad \text{(B13)}$$

The noise spectrum is bounded above on Ψ, so the maximum in (B13) is finite. Thus, d_{Eckart} is made arbitrarily large by choosing ϵ sufficiently small. In other words, the most detectable signals are those whose spectra contain at least one component of finite power and very narrow bandwidth.

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